Convolution of a Rectangular "Pulse" With Itself

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After failing in my attempts to locate online a derivation of the convolution of a general rectangular pulse with itself, and not having available a textbook on communications or signal processing theory, I decided to write up my attempt at computing it. I expect, however, that it is the first example one would find in any textbook that discusses convolution.

Recall the general definition of the convolution $f * g$ of two real-valued functions:

$$(f * g)(t) \;=\; \int_{-\infty}^{\infty} f(u)\, g(t-u)\, du \;=\; \int_{-\infty}^{\infty} f(t-u)\, g(u)\, du. \tag{1}$$

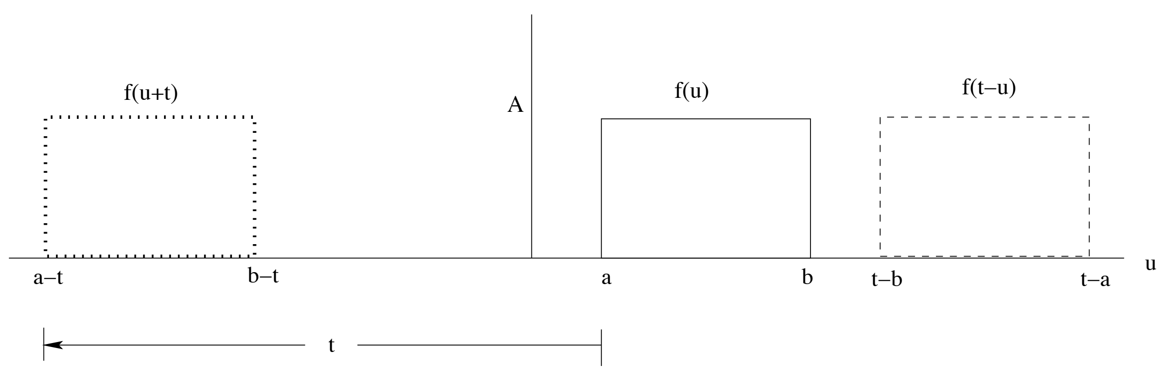
We apply this to the problem where f and g are both given by

$$f(t) \;=\; g(t) \;=\; \begin{cases} 0, & t < a, \\ A, & a \le t \le b, \\ 0, & t > b, \end{cases} \tag{2}$$

where $[a, b]$ is a time interval on the real line, with $a < b$. In signal processing this represents a rectangular pulse of amplitude A and width or duration $T = b - a$. The convolution of this function with itself is the time-dependent function

$$(f * f)(t) \;=\; \int_{-\infty}^{\infty} f(u)\, f(t-u)\, du. \tag{3}$$

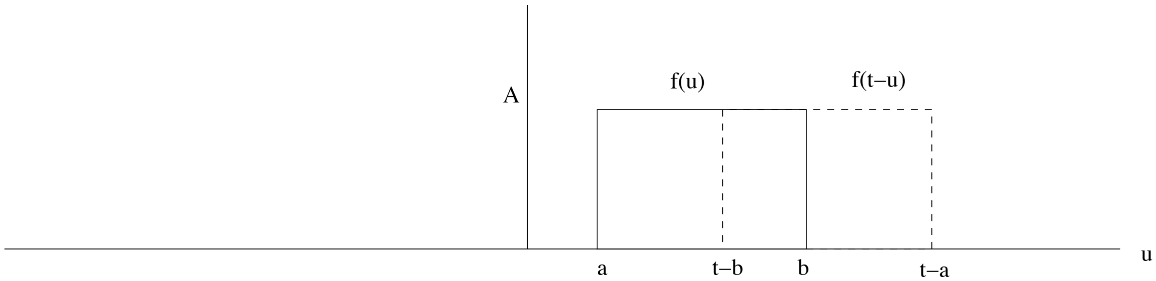
How is $f(t-u)$ related to $f(u)$? Define $g(u) = f(u+t)$, which for $t > 0$ represents a horizontal translation of $f(u)$ to the *left* by t. Then $h(u) = g(-u) = f(-u+t) = f(t-u)$ is a *reflection* of $g(u)$ across the vertical axis $u = 0$. Thus, as a function of u, $f(t-u)$ is a replica of $f(u)$ which, for $t > 0$, is *translated to the left* a distance t, then *reflected* across the vertical axis $u = 0$. Thus, $f(t-u)$ is a function whose values depend on both u and t. The convolution is calculated for each value of t on the real line by integrating over the real line (with respect to u) the *product* of the two functions $f(u)$ and $f(t-u)$ at that value of t. For the rectangular function defined by (2), graphs of the functions $f(u)$, $f(u+t)$, and $f(t-u)$ for $t > 0$ are illustrated below:



Referring to the Figure, observe that as the distance t from a increases, the translated and reflected pulse $f(t-u)$ moves to the *right* toward $+\infty$. On the other hand, as the distance t to the left of a decreases, the *translated* pulse $f(u+t)$ moves toward the *right*, and the translated and reflected pulse $f(t-u)$ moves toward the *left*.

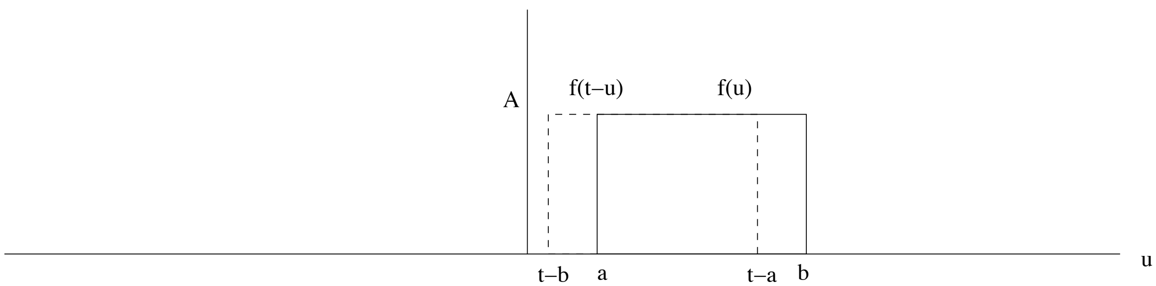
Focusing first on the *left* edge $u = t-b$ of $f(t-u)$ (represented in the next two Figures by the rectangle with "dashed line" sides), we see that for $u = t-b > b$, the original pulse has value $f(u) = 0$, so the convolution will be zero for $t > 2b$, corresponding to $u > b$.

When $a < t-b \le b$, or $a+b < t \le 2b$, both $f(u) = A$ and $f(t-u) = A$. But $f(t-u) = 0$ for $u \le t-b$, so the integrand is nonzero only for $t-b \le u \le b$, as shown in this Figure:



The two pulses coincide exactly when $t-b = a$, and $t-a = b$. that is, when $t = a+b$.

For $t > a+b$, we focus on the *right* edge $u = t-a$ of $f(t-u)$ as it moves thru $f(u)$ to the left. For $a \le t-a < b$, or $2a \le t < a+b$, both $f(u)$ and $f(t-u)$ have amplitude A, but $f(t-u) = 0$ for $u > t-a$, hence the integrand is nonzero only for $a \le u \le t-a$. The situation is illustrated in this Figure:

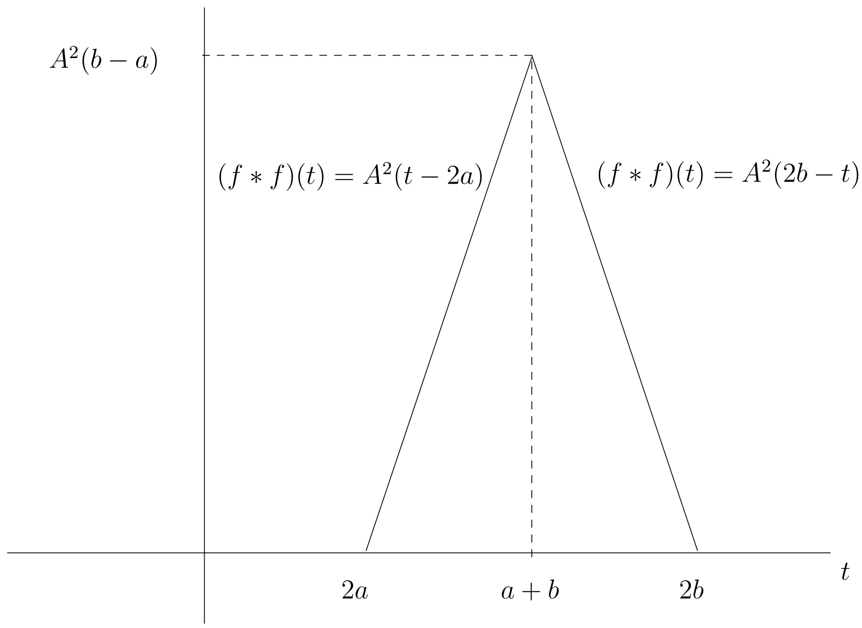


Finally, for $u = t-a < a$, or $t < 2a$, the original pulse $f(u) = 0$, so the convolution is again zero for $u < a$.

These results are summarized in the following calculations of the convolution for any value of t, listed in the opposite order of the above discussion:

$$\begin{cases} (f*f)(t) \;=\; 0, \qquad \text{for} \qquad t \;<\; 2a, \\[2ex] (f*f)(t) \;=\; \displaystyle\int_a^{t-a} A\cdot A\,du \;=\; A^2\,(t \;-\; 2a), \qquad \text{for} \qquad 2a \le t \le a+b, \\[2ex] (f*f)(t) \;=\; \displaystyle\int_{t-b}^{b} A\cdot A\,du \;=\; A^2\,(2b \;-\; t), \qquad \text{for} \qquad a+b \le t \le 2b, \\[2ex] (f*f)(t) \;=\; 0, \qquad \text{for} \qquad t \;>\; 2b. \end{cases} \tag{4}$$

The graph of this piecewise-defined function is an *isosceles triangle* of height $A^2\,(b\,-\,a)$ at the vertex point $(\,(a+b), A^2(b-a)\,)$, and base of width $2(b\,-\,a)$ with vertices at the points $(2a, 0)$ and $(2b, 0)$ on the t–axis. These details are illustrated in the next Figure.



$A^2(b-a)$
$(f*f)(t) = A^2(t-2a)$
$(f*f)(t) = A^2(2b-t)$
$2a$
$a+b$
$2b$
t